An invariant approach to gauge theory via coupled Dirac equation

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Abstract

We propose a strategy to study gauge theory in a gauge invariant fashion by considering the coupled Dirac equation. We give several motivations of this approach, from quantum mechanics, algebraic geometry, the ADHM construction, the G2 instanton equation, and analytical considerations. The general picture, which is partly conjectural, is that under some smallness assumptions on curvature, the (possibly non-smooth and not necessarily Yang-Mills) connnections along with the bundle can be recovered from certain solutions to the coupled Dirac equation with natural additional data, which satisfy bounds depending only on the norm of the curvature.

We explain a possible strategy and the difficulties in this approach. Then we address certain analytic issues, in particular, how to bound the solutions of the Dirac equation knowing only some fairly weak smallness bounds on curvature. We use that to study the convergence behaviour of Dirac fields associated with a sequence of connections under uniform small curvature bounds. Then we describe a variational approach to construct Dirac fields with required bounds. Under some assumptions about the maximum value associated to the variational problem, we prove a rigidity theorem which bounds all quantities of interest under the small curvature assumption.

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Chapter 1

Preliminaries

1.1 Gauge theory background

Gauge theory is concerned with the study of connections A and curvatures F on principal bundles and their associated vector bundles over a manifold M, often taken to be compact. Of particular interest is the concept of Yang-Mills connections, which are critical points of the Yang-Mills energy functional

$$YM(A) = \int_M |F|^2 dVol$$

The Euler-Lagrange equation for this functional is called the Yang-Mills equation. A crucial feature of the Yang-Mills functional is that it is gauge invariant; in particular, a gauge transform sends a Yang-Mills connection to another Yang-Mills connection.

Prominent types of gauge theories study Yang-Mills connections under special geometric assumptions on the manifold. What typically happens is that the Yang-Mills functional can be written as a topological term plus a positive quadratic term, which enables us to characterise the absolute minima of the Yang-Mills equation in terms of first order differential equations.

Example 1.1.1. (Anti-self-dual connections) Consider a Hermitian vector bundle on a compact oriented 4-dimensional manifold. The curvature decomposes into the self-dual and anti-self-dual parts F^+ and F^-. We have the topological identity

$$YM(A) = \int_M |F^+|^2 dVol + \int_M |F^-|^2 dVol = 2\int_M |F^+|^2 dVol - 8\pi^2 c_2(E)$$

where $c_2(E)$ is the second Chern class, which we assume is negative. Hence the Yang-Mills functional has a topological bound which is achieved precisely when $F^+ = 0$ (the ASD equation). (cf. Pg 40, (Donaldson and Kronheimer)).

Example 1.1.2. (Hermitian-Yang-Mills connections on a complex surface) On a Kähler surface with Kähler form ω, the ASD condition takes especially simple form:

$$\begin{cases} F^{0,2} = 0 \\ F \wedge \omega = 0 \end{cases}$$

This is called the Hermitian-Yang-Mills equation. The first equation means the connection is complex integrable, and therefore defines a holomorphic bundle structure; this integrability is crucial for the algebraic geometric methods to apply. (cf. Chapter 2, (Donaldson and Kronheimer)).

Example 1.1.3. (G2 instantons) On a G2 manifold with defining 3-form ϕ, the 2-forms decompose into a 7D representation and a 14D representation of the holonomy group. This induces a decomposition on curvature $F = F_{(7)} + F_{(14)}$, which leads to the topological identity

$$YM(A) = \frac{1}{3}\int_M |F + *(F \wedge \phi)|^2 dVol - 8\pi^2 p_1(\mathfrak{g}_E) \cup [\phi].$$

From this one obtains a topological lower bound on the Yang-Mills energy, which is achieved precisely when

$$F_{(7)} = 0.$$

Solutions of this equation are called G2 instantons. (cf. Pg 31, (Walpuski)).

The techniques of gauge theory usually involves embedding the solutions of interest into a suitable Banach space and then linearise the equation of interest near a given solution. To fit this procedure into the framework of elliptic PDEs, one needs a gauge fixing procedure which removes the gauge invariance of the equation.

Remark. For us this means two things: first, even if one is interested only in the exact solutions of the Yang-Mills equation (or the first order reductions) one is obliged to consider connections in a Banach space. Second, gauge invariance is lost in the conventional approach.

1.2 The coupled Dirac equation

Let's recall the second piece of background. We consider a connection on a Hermitian bundle E coupled to the spinor bundle S on a general Riemannian spin manifold, which gives rise to the Dirac operator on the bundle $E \otimes_{\mathbb{C}} S$. Alternatively, one can couple an $O(n)$ vector bundle E to the spinor bundle and study the Dirac operator on $E \otimes_{\mathbb{R}} S$. These two cases are very similar and our treatment will be mostly uniform. The Weitzenbock formula provides the basic link between the Dirac operator and the Laplacian. We review the setup:

Let e_i be an orthonormal frame of the tangent bundle, for i from 1 to n. These act as Clifford multiplications c_i on the spin bundle, satisfying the Clifford relations and the anti-self-adjoint relations:

$$c_i c_j + c_j c_i = -2\delta_{ij},\ c_i = -c_i^\dagger.$$

With respect to this frame, the Levi-Civita connection takes the form

$$\nabla_j = \partial_j + A_j,\ A_j = (a^i_{jk} e_i \otimes e_k^*)$$

where $a^i_{jk} = -a^k_{ji}$, so the connection matrix is $A_j = \frac{1}{2} a^i_{jk}(e_i \otimes e_k^* - e_k \otimes e_i^*)$. The Riemannian curvature is

$$Riem = \nabla^2 = \frac{1}{2} R_{(ij)} e_i^* \wedge e_j^*,\ R_{(ij)} = \nabla_i \nabla_j - \nabla_j \nabla_i - \nabla_{[e_i, e_j]}$$

The Levi-Civita connection induces a connection on the spin bundle, by regarding the spin bundle as the bundle associated to the spin representation of the structure group $SO(n)$. The spin representation at the level of the Lie algebra $so(n)$ is given by:

$$\rho : e_i \otimes e_j^* - e_j \otimes e_i^* \mapsto -\frac{1}{2} c_i c_j.$$

Then the spin connection is given by $\nabla_j = \partial_j + \rho(A_j)$. The curvature of this connection is $\frac{1}{2}\rho(R_{(ij)}) e_i^* \wedge e_j^*$.

Now if we have a Hermitian bundle equipped with a unitary connection ∇, or an $O(n)$ vector bundle with an orthogonal connection, then there is an induced connection on the tensor product bundle $E \otimes S$, still denoted ∇, and we can define the Dirac operator $D = c_i \nabla_i$. It is well known to be self adjoint. The connection operator $\nabla : \Gamma(E \otimes S) \to \Gamma(E \otimes S \otimes T^*M)$ also has an adjoint $\nabla^* = -\nabla_j \iota_{e_j}$ where ι denotes the inner contraction with a tangent vector. So the Laplacian term is $\nabla^*\nabla = -\nabla_j \nabla_j$. This can be compared with the square of the Dirac operator using the fundamental Weitzenbock formula:

Theorem 1.2.1. *(Weitzenbock formula)*

$$D^2 = \nabla^*\nabla + \frac{1}{4} R + \sum_{i<j} c_i c_j F_{ij} \tag{1.1}$$

where R is the Ricci scalar and F is the curvature of the connection on E, $F = \frac{1}{2} F_{ij} e_i^* \wedge e_j^*$. It is convenient to denote $\tilde{F} = \sum_{i<j} F_{ij} c_i c_j$.

Proof. We compute in a frame where $\nabla e_i = 0$.

$$\begin{aligned} D^2 &= \sum_{i,j} c_i \nabla_i c_j \nabla_j = \sum c_i c_j \nabla_i \nabla_j \\ &= -\nabla_j \nabla_j + \sum_{i<j} c_i c_j (\nabla_i \nabla_j - \nabla_j \nabla_i) \\ &= \nabla^*\nabla + \sum_{i<j} c_i c_j [\rho(R_{(ij)}) + F_{ij}] \end{aligned}$$

It remains to simplify the curvature terms.

$$R_{(ij)} = -R_{ijkl} e_k \otimes e_l^* = -\frac{1}{2} R_{ijkl} (e_k \otimes e_l^* - e_l \otimes e_k^*)$$

$$\rho(R_{(ij)}) = \frac{1}{4} R_{ijkl} c_k c_l$$

Now

$$\sum_{i<j} c_i c_j \rho(R_{(ij)}) = \frac{1}{4} \sum_{i<j} c_i c_j c_k c_l R_{ijkl}.$$

Using the Bianchi identity, the case of $k \notin \{i, j\}$ contributes zero to the sum. Thus

$$\sum_{i<j} c_i c_j \rho(R_{(ij)}) = -\frac{1}{4} (\sum_{i<j} \sum_l c_i c_l R_{ijjl} + \sum_{i<j} \sum_l c_i c_l R_{jiil}) = -\frac{1}{4} c_i c_l R_{ijjl} = \frac{1}{4} R_{ijji} = \frac{1}{4} R$$

from which the Weitzenbock formula follows. □

Corollary 1.2.2. Let Δ denote the Hodge Laplacian on scalar functions. Then for a solution s of the Dirac equation (which we call a Dirac field), we have

$$-\Delta|s|^2 = 2|\nabla s|^2 + 2\langle (R/4 + \tilde{F})s, s\rangle. \tag{1.2}$$

Proof. We compute using Weitzenbock formula:

$$\begin{aligned}
-\Delta|s|^2 &= \nabla_i \nabla_i |s|^2 = 2\nabla_i Re\langle \nabla_i s, s\rangle \\
&= 2|\nabla s|^2 + 2Re\langle \nabla_i \nabla_i s, s\rangle \\
&= 2|\nabla s|^2 + 2\langle (R/4 + \tilde{F})s, s\rangle.
\end{aligned}$$

□

1.3 Motivations for the study of Dirac equation

There are at least five perspectives from which the study of the coupled Dirac equation is interesting for understanding gauge theory, especially if we want to approach gauge theory without a traditional gauge fixing procedure.

1.3.1 Viewpoint of physics

According to quantum mechanics, the Dirac equation describes the motion of fermions inside the gauge field. This is a question of independent interest, and historically is where the Dirac equation was discovered. From the viewpoint of gauge theory the importance of the Dirac equation comes from the fact that the quantum mechanical observables for solutions of the Dirac equation will provide gauge invariant quantities for the connection.

More precisely, suppose s and t are solutions of the Dirac equation. Then the quantum mechanical observable $\langle s, c_i t\rangle$ is closely related to the charge current, and the observable $\langle s, \nabla t\rangle$ is closely related to the energy-momentum vector. These quantities are defined in a gauge invariant way. Morever we expect to be able to reconstruct the connection once we have enough observables of that kind. The physical meaning of this is clear, namely that to detect the presence of a gauge field one had better to observe the motion of fermions in these fields.

1.3.2 Viewpoint of Algebraic Geometry

Spin geometry is the analogue of holomorphic geometry when we work in the Riemannian world. The formal similarity is indicated by the fact that in the absence of extra connections, then the Dirac operator is a first order operator acting as 'the square root of the Laplacian', which is the content of the Weitzenbock formula; this is an analogue of the well known relations between the Laplacian and the $\bar{\partial}$ operator on a Kähler manifold. The difference is that the Dirac operator exists in much greater generality and does not depend on special integrability conditions.

In algebraic geometry, the connections of interest are Chern connections on holomorphic vector bundles. The (0,1) part of the Chern connection specifies the holomorphic structure of the bundle; conversely the Hermitian condition and compatibility with the holomorphic structure pins down the Chern connection. It is also conventional to think of the holomorphic bundle in terms of the sheaf of holomorphic sections. So the upshot is that if we fix a Hermitian structure on a complex vector bundle, then giving the Chern connection is the same as giving the sheaf of holomorphic sections, endowed with the inner product structure. We note that the sheaf theoretic description is naturally gauge invariant.

Now by analogy, we hope for the similar picture to be true in spin geometry, namely that giving a connection on a Hermitian vector bundle E should be roughly equivalent to giving the 'sheaf of solutions to the Dirac equation'. The solutions of the Dirac equation are endowed with extra structures, such as inner products, Clifford multiplications and gradients.

We also remark that if we look for holomorphic vector bundles in families, then we will naturally need the concept of coherent sheaves in order to take limits of the vector bundles. One may wonder what is the correct analogue in the spin geometric world. This question is related to compactification problems for the space of connections.

1.3.3 Relevance to the ADHM construction

The ADHM construction gives the ASD connections on the flat Euclidean $\mathbb{R}^4$ in terms of the ADHM data. A detailed account is given in Chapter 3 of (Donaldson and Kronheimer). For motivational purposes, it suffices to remark that the way to obtain the ADHM data from an ASD connection is by considering the finite dimensional space of global solutions to the coupled Dirac equation satisfying growth conditions, some bilinear products related to the Clifford multiplication, together with linear operators related to the evaluation of these Dirac fields at the fibre at infinity.

1.3.4 Relevance to G2 manifolds

One of the characterisation of G2 manifolds is that there exists a global parallel spinor field. This induces a decomposition of the spin bundle into a trivial factor, plus a part isomorphic to the tangent bundle:

$$S = \mathbb{R} \oplus TM.$$

This suggests that coupling a G2 instanton (and its possible perturbations) to the spinor field may provide a way to explore the G2 condition.

From yet another viewpoint, the spin geometry enters into the theory of G2 instantons via associative submanifolds. In (Donaldson and Segal) it was argued that associative submanifolds should arise from limits of G2 instantons; more precisely, a compactness argument implies that a sequence of G2 connections will converge smoothly away from some singular locus of Hausdorff dimension at most 3, and one expected type of singularity formation is for the energy density $|F|^2$ to blow up along an associative submanifold. But according to (Mclean) the first order deformation of the associative submanifold is governed by a twisted Dirac equation along the associative submanifold, by suitably interpreting the normal bundle to the associative submanifold as a spin bundle.

What one might hope for, is that by thinking of the G2 instantons in terms of the sheaf of solutions to the Dirac equation, one may be able to unify the connections with the associative submanifolds in a natural fashion, much as the picture in algebraic geometry where the viewpoint of sheaves unify vector bundles with subvarieties. Technically this picture is difficult to achieve because it requires us to understand the non-perturbative effects of curvature, namely that we can no longer impose a small energy assumption.

1.3.5 Analytic motivations for the Dirac equation

As remarked at the end of 1.1 it is necessary in gauge theory to consider connections which are not necessarily Yang-Mills, but serve as perturbations to the Yang-Mills connections. Then one is faced with the question of choosing the normed space to work with. The general feature of the elliptic equations poses the following dilemma:

If we work with a strong normed space, then the elliptic estimates are easier, and this is good enough for many purposes. But we usually cannot expect compactness results, and the norms will not be controlled by energy.

If we work with a weak normed space, such as imposing the assumption $\|F\|_{M_2^{n/2}} < \infty$, which is natural in the light of Price monotonicity formula, then the elliptic estimates are hard. Morever, there will be technical difficulties arising from the failure of Sobolev embedding theorems; the gauge transforms with too little regularity will not be able to compose, so the gauge group cannot be defined.

If we insist on doing everything invariantly, then the problem is more serious, because the Yang-Mills equation and the first order reductions are not elliptic. However, the Dirac equation is elliptic, a fact which strongly motivates the analytic study of Dirac equation as a gauge invariant approach to gauge theory. Also, because the Dirac equation is gauge invariant, the problem for the failure of defining the gauge group does not arise. This means in principle working with the Dirac equation may allow us to bypass a number of analytic difficulties which arises from gauge invariance and the desire of very weak assumptions on curvature.

1.4 More on analytic aspects

Good background readings about analytical aspects of gauge theory can be found in (Donaldson and Kronheimer), and higher dimensional analytic tools are collected in (Tian and Tao). Here we point out some crucial issues necessary for the motivation of our work on Dirac equation. In the sequel the dimension is at least 4.

1.4.1 Price monotonicity formula

The Price monotonicity formula is a fundamental result for Yang-Mills connections in higher dimensional gauge theory ($n = dim(M) \geq 4$), and a good account is given in Chapter 2, (Tian). For simplicity we state it in the Euclidean case.

Theorem 1.4.1. *For a Yang-Mills connection on Euclidean* $\mathbb{R}^n$*, and* $0 < \sigma < \rho$*, we have*

$$\rho^{4-n} \int_{B(\rho)} |F|^2 dVol - \sigma^{4-n} \int_{B(\sigma)} |F|^2 dVol \geq 4 \int_{B(\rho)\setminus B(\sigma)} r^{4-n} |\frac{\partial}{\partial r} \lrcorner F|^2 dVol$$

Upshot: The curvature separates into the radial and the perpendicular part. $F = F^{\parallel} + F^{\perp}$, where $F^{\parallel} = dr \wedge (\frac{\partial}{\partial r} \lrcorner F)$. The radial part of the curvature is less singular than the perpendicular curvature.

1.5 The main picture

1.5.1 Statement of the goal

The ultimate goal is to justify the picture sketched in 1.3.2. We state it in perhaps not the strongest way.

Statement 1.5.1. (Goal) Suppose the curvature F is small in a suitable sense. Let the possibly singular connection A be the limit of a sequence of smooth connections $A_{(i)}$, converging say in the L^2_1 norm. The curvature $F_{(i)}$ are required to satisfy uniform small bounds. Then there are $m = rkE$ solutions to the Dirac equation $D_A(s) = 0$, denoted s_j, $j = 1, 2, \ldots, m$, defined on a neighbourhood of some definite size, which satisfy the additional requirements that

- The individual s_j are close to being pure tensors $s'_j = \xi_j \otimes \eta_j$ in $E \otimes S$, in the sense of sup estimates, where s'_j are obtained from the Dirac fields using some definite formula involving only zeroth order observable quantities.
- Zeroth order quantities, *i.e.* bilinear expressions defined by Clifford multiplications, are required to be continuous.
- The E bundle component of the pure tensor fields, ξ_j are close to being orthonormal, in the sense of sup estimates.
- The gradient of the fields s_j are small in the sense of integral estimates.

Morever the data of Clifford multiplication, the gradient and the inner product structure are sufficient to reconstruct the connection.

Remark. This collection of requirements is what we mean by saying the sheaf of solutions to the Dirac equation resembles the sheaf of sections of a vector bundle, with ξ_j defining a local basis of the vector bundle. An even stronger statement may be made by weakening the sense of convergence, for example by allowing the curvature to converge in the sense of measures. If we can weaken the conditions to this level, then the statement will allow us to probe compactification questions.

1.5.2 The strategy

We outline the strategy to prove the statement. So far I still need an additional spectral assumption (*cf.* chapter on construction of Dirac fields). The heart of this is to derive uniform estimates.

1. We need a characterisation of pure tensor fields in terms of zeroth order quantities. If this characterisation is approximately satisfied, then we need a formula to perturb the sections into pure tensor fields, and a reconstruction formula for the connection.

2. Impose a possibly small uniform bound on curvature. This is chosen at our discretion, but ideally this bound is controlled by terms appearing in Price monotonicity formula.

3. For smooth connections with this bound, prove the existence of Dirac fields, defined on some ball whose size depend only on the Riemannian geometry and the bound on curvature. We require the sections to be approximately pure tensor fields whose E bundle component are required to be almost orthonormal, and the S components are C^0 close to parallel spinor fields.

   Remark. This step is hard. However, if no uniform estimates are required, this may be done quite easily, because Dirac operators are small perturbations of the standard one in a sufficiently small ball.

4. Derive for these Dirac fields uniform sup estimates and gradient estimates depending only on the Riemannian geometry and the curvature bound. Derive uniform estimates on the zeroth order quantities. Ideally obtain equicontinuity estimates.

5. Use some compactness results to take subsequential limits for the section, or at least for the observable quantities. Justify that the limit sections satisfy the Dirac equation for the limiting connection. Justify that norms will not collapse. Justify that zeroth order quantities are continuous when we pass to the limit.

Remark. The sup estimate is strictly necessary for the limiting data to define some objects resembling a topological vector bundle.

Having said these, working with very weak assumptions on curvature is difficult. Therefore in this report we will in general make smallness assumptions on curvature. At the moment I am also unable to work with the optimal norm on the curvature. The assumptions I use in the later work are worse than the critical Morrey norm $\|F\|_{M_2^{n/2}}$ but weaker than the norms $\|F\|_{M_2^p}$ where $p > n/2$.

1.6 Reconstruction of connections

We discuss how to reconstruct the smooth connection from gauge invariant quantities of the Dirac fields. This is the easiest step in the whole strategy.

We notice another piece of structure: the bundle $E \otimes S$ tensoring S has a contraction map into E. This clearly does not depend on any gauge choices. We first use this to characterise the pure tensor fields.

Proposition 1.6.1. A tensor s of unit norm inside $E \otimes S$ can be written as $\xi \otimes \eta$, where η is a unit spinor in S, if and only if the contraction of s with η, denoted $\langle \eta, s \rangle$, has norm 1.

Remark. We notice $|\langle \eta, s\rangle|^2 = \langle s|\eta^\dagger \eta|s\rangle$ using quantum mechanical notations, where $\eta^\dagger \eta$ is some Clifford multiplication operator. So this characterisation is purely numerical in nature, and only uses the inner product structure and the Clifford multiplications. We may also notice that the same can be said of the orthonormality condition mentioned in the last section. In particular the zeroth order numerical observables $\langle s|\text{Clifford}|s\rangle$ reconstruct the metric on the bundle.

Once we have this characterisation, it is clear that for an approximate pure tensor, we can construct $\xi = \langle \eta, s\rangle$, $s' = \xi \otimes \eta$, so that $|s - s'|$ has a small bound. This means we know how to perturb an approximate pure tensor into a genuine pure tensor.

With these preparations we can reconstruct the connection. Take the setup in the last section on the main picture. Then ξ_j gives an approximate local orthonormal basis of sections of E. We demand the η_j to have small gradients; in the Euclidean case, we may just take the η_j to be the same specific parallel spinor field. The contraction map satisfies

$$\nabla_k \langle \eta, s\rangle = \langle \nabla_k \eta, s\rangle + \langle \eta, \nabla_k s\rangle$$

The information of the connection is encoded by the connection coefficients $\langle \xi_i, \nabla_k \xi_j\rangle$. For clarity, let's talk about the Euclidean case, where $\nabla \eta = 0, \eta_j = \eta$. Then the connection coefficients are just

$$\langle s_i|\eta^\dagger \eta|\nabla_k s_j\rangle$$

using the quantum mechanical notations. This means:

Proposition 1.6.2. Under the assumption of the last section, the connection coeffients can be reconstructed from the data of Clifford multiplication, gradient and inner product structure on the Dirac fields. Morever, smallness bounds on the Dirac fields and the gradients easily lead to bounds on the connection coefficients.

Remark. In case we can achieve bounds on the sup norm of s, and the Morrey norm $\|\nabla s\|_{M_2^{n/2}}$, then the connection coefficients have an estimate in the $M_2^{n/2}$ sense. This means our picture in particular would imply an Uhlenbeck gauge fixing type result, despite the gauge invariance of the argument.

Chapter 2

A priori estimates on the Dirac fields

The goal of this chapter is to derive estimates of the following nature:

Theorem 2.0.3. *Consider the coupled Dirac equation for a smooth connection. Assume $Ds = 0$ on a ball $B(2R)$ in the Riemannian manifold of dimension at least 4, with radius smaller than the injectivity radius, and $|Riem|R^2 << 1$. Then under suitable smallness assumptions on curvature, to be specified later, we have sup bounds of the form*

$$|s|(y) \leq \frac{C}{r^{n/2}} \|s\|_{L^2(B(y,r))} \tag{2.1}$$

for $B(y,r) \subset B(2R)$, and the interior gradient estimate

$$\int_{B(y,r/2)} (\frac{1}{|x-y|})^{n-2} |\nabla s|^2(x) dVol \leq \frac{C}{r^n} \|s\|^2_{L^2(B(y,r))} \tag{2.2}$$

where the constant only depends on the Riemannian geometry and some small bounds on curvature.

We first do so using a subharmonicity estimate coming from the Weitzenbock formula, and then refine the estimate by a careful separation of assumptions on the radial and perpendicular components of the curvature. This separation is motivated by Price monotonicity formula, which morally says that the radial curvature has more regularity. More precise conditions are given in 2.2.2 and 2.4.1.

2.1 Estimate based on subharmonicity

Consider the inequality (actually the equality holds)

$$(-\Delta)|s|^2 \geq 2|\nabla s|^2 + 2\langle \tilde{F}s, s\rangle$$

where the sign convention of the Hodge Laplacian is given by $\Delta = \sum \partial_i^2$ in the Euclidean case. This comes from the Weitzenbock formula, assuming $Ds = 0$ and the Ricci scalar to be 0. The assumption on Ricci scalar is satisfied on a G2 manifold, and in the general case does not affect the main estimates.

Let $u = |s|^2$. Notice the gradient term is the dorminant forcing term, which has the positive sign, so we expect the solution to behave like a subharmonic function, which means a Harnack type inequality should be satisfied. We aim to derive estimates on the Dirac field s which depend only on the Riemannian geometry and some Morrey type norms of the curvature F with critical exponent. The consideration of critical exponent is natural from the viewpoint of Price monotonicity formula, but technically makes the estimates much harder. In particular, a naive application of Moser iteration method does not seem to work in this context.

It seems easiest to explain the ideas in the Euclidean setting. We will later briefly remark on the Riemannian corrections. We begin by deriving a version of mean value property following a standard approach.

To measure the subharmonicity effect, we introduce a weighted average of spherical means

$$g(t) = \frac{1}{t^n} \int_{B(tR)} u(x) dVol(x) = \int_{B(R)} u(tx) dVol(x). \tag{2.3}$$

We compute its derivative

$$\frac{dg}{dt} = \frac{1}{t^{n+1}} \int_0^{tR} r dr \int_{|x|=r} \frac{\partial u}{\partial \vec{n}}(x) dA(\theta, r) \tag{2.4}$$

where $dA(\theta, r)$ is the polar area element, and θ is the angular variable. This essentially used the dilational homogeneity of the Lebesgue measure. In the more general Riemannian case, we need the formula for the variation of area element along the geodesics, which uses the asymptote of the mean curvature of the geodesic spheres. This will give a correction term of order

$$|Riem| \frac{1}{t^{n+1}} \int_0^{tR} r^2 dr \int_{|x|=r} u(x) dA(\theta, r).$$

We will always assume the radius of the ball to be less than the injectivity radius, and $|Riem| R^2 << 1$.

Notice the boundary integral can be converted into the volume integral of the Laplacian.

$$\int_{|x|=r} \frac{\partial u}{\partial n}(x) dA(\theta, r) = \int_{|x|\leq r} -\Delta u \mathrm{dVol(x)}. \tag{2.5}$$

We proceed to exploit the positivity of the dorminant term in the derivative of the function g. In the Euclidean case, the RHS of 2.4 is bounded below by:

$$\frac{dg}{dt} \geq \int_0^{tR} r dr \int_{|x|\leq r} [2|\nabla s|^2 + 2\langle \tilde{F} s, s\rangle dVol(x)] \frac{1}{t^{n+1}} \tag{2.6}$$

Now we integrate in t from 0 to 1 to get:

$$\begin{aligned} g(1)-g(0) &\geq \int_0^1 dt \int_0^{tR} rdr \int_{|x|\leq r} [2|\nabla s|^2 + 2\langle \tilde{F}s, s\rangle dVol(x)] \frac{1}{t^{n+1}} \\ &= \int_{B(R)} [\frac{1}{n(n-2)} (\frac{R}{r})^{n-2} R^2 - \frac{1}{2(n-2)} R^2 + \frac{1}{2n} r^2] \times \\ &\quad [2|\nabla s|^2 (r\theta) + 2\langle \tilde{F}s, s\rangle] dVol \end{aligned} \tag{2.7}$$

Remark. One can think of this as the analogue of Green's formula for the Dirichlet problem of the Laplace equation. The point is that $g(0)$ is essentially the value of $|s|^2$ at the origin, on which this estimate puts an upper bound.

Notice the gradient term is multiplied by a positive expression, which tends to 0 at the boundary $r = R$, and remains bounded below away from the boundary. We remark also that the non-Euclidean effect is bounded by

$$C|Riem| \int_{B(R)} (\frac{R}{r})^{n-2} |s|^2 dVol$$

which is easily dealt with by Hardy's inequality 2.5.2. Combining the above immediately leads to:

Proposition 2.1.1. We have the subharmonicity estimate

$$\begin{aligned} &|s(0)|^2 Vol(B(R)) + \int_{B(R/2)} (\frac{R}{r})^{n-2} R^2 |\nabla s|^2 (r\theta) dVol \\ &\leq C[\|s\|^2_{L^2(B(R))} + R^2 \int_{B(R)} |\langle \tilde{F}s, s\rangle| (\frac{R}{r})^{n-2} dVol]. \end{aligned} \tag{2.8}$$

2.2 The sup norm estimate

We aim to bound $\|s\|_{L^\infty}$. For clarity, let's ignore non-Euclidean complications. This is justified because non-Euclidean correction terms have milder singularity behaviour. Since we are working with smooth solutions, we can assume for any $y \in B(2R), B(y,r) \subset B(2R)$, we have an estimate of the form

$$|s|(y) \leq \frac{M}{r^{n/2}} \|s\|_{L^2(B(y,r))}. \tag{2.9}$$

The task is to show M is independent of s. The main idea is that under some small energy assumption, after a chain of estimates, we will be able to improve the ad hoc constant M. This argument is in the spirit of a maximum principle.

For this we need to impose some bound on the curvature F. Put

$$\|F\|_1 = \sup_{B(y,r) \subset B(2R)} (\int_{B(y,r)} \frac{|F|(x)}{|x-y|^{n-2}} dx) \tag{2.10}$$

The assumption is that $\|F\|_1$ is small.

Since we have the pointwise bound on $|s|$ inside the ball $B(R)$, the following estimate is immediate:

Lemma 2.2.1. *We have*

$$R^2 \int_{B(R)} |\langle \tilde{F}s, s\rangle| (\frac{R}{r})^{n-2} dVol \leq M^2 \|s\|^2_{L^2(B(2R))} \|F\|_1$$

Combining this with the subharmonicity estimate 2.1.1, one obtains

$$(2R)^n |s|^2(0) \leq C(1 + M^2 \|F\|_1) \|s\|^2_{L^2(B(2R))}$$

where C is independent of M, R. More generally, this argument works for any point inside $B(2R)$; if $B(y,r) \subset B(2R)$, then

$$r^n |s|^2(y) \leq C(1 + M^2 \|F\|_1) \|s\|^2_{L^2(B(y,r))}$$

Now comes the **crucial observation**: The constant M can be changed to $C(M \|F\|_1^{1/2} + 1)$, which still gives a bounding constant. So if M is the optimal constant, then $M \leq C(M \|F\|_1^{1/2} + 1)$. But now if $\|F\|_1^{1/2} C < 1$, then this implies $M \leq C$, where this changed constant C is now independent of s. To summarise,

Theorem 2.2.2. *Under the assumption that $\|F\|_1$ is small, we have the estimates*

$$|s|(y) \leq \frac{C}{r^{n/2}} \|s\|_{L^2(B(y,r))} \tag{2.11}$$

and the interior gradient estimate

$$\int_{B(y,r/2)} (\frac{1}{|x-y|})^{n-2} |\nabla s|^2(x) dVol \leq \frac{C}{r^n} \|s\|^2_{L^2(B(y,r))}. \tag{2.12}$$

2.3 Integral identities

A more refined estimate can be obtained by a delicate separation of assumptions on the curvature term, which is natural from the perspective of the Price monotonicity formula, namely that the radial part of curvature should be less singular than the perpendicular part.

In this section, we prepare a number of integral identities which will be useful later. To motivate these preparations, one can have the following analogy in mind: the Cauchy-Riemann equation is a first order equation which contains more refined information than the Laplace equation. Similarly the Dirac equation has more information than the Laplace equation. To extract such information requires us to go beyond the Weitzenbock formula.

2.3.1 Differential identities

Recall that the curvature decomposes into the radial and the perpendicular part: $F = F^{\parallel} + F^{\perp}$, where $F^{\parallel} = dr \wedge (\frac{\partial}{\partial r} \lrcorner F)$. We want to extract from the Dirac equation certain effects which depend only on the radial curvature $F^{\parallel}$, on which we can impose slightly stronger conditions than the conditions on F.

To this end, it is advantageous to introduce some operators and study their commutation rules. Let the operator $\tilde{dr}$ be the Clifford multiplication on the spinors by the tangent vector ∇r. We have $(\tilde{dr})^2 = -1$. In the Euclidean case, $\tilde{dr} = \frac{x_i c_i}{|x|}$.

Lemma 2.3.1. $\tilde{F}\tilde{dr} - \tilde{dr}\tilde{F} = 2\tilde{F^{\parallel}}\tilde{dr}$

Proof. We have $\tilde{dr}\tilde{F} = \tilde{dr}\tilde{F^{\parallel}} + \tilde{dr}\tilde{F^{\perp}} = -\tilde{F^{\parallel}}\tilde{dr} + \tilde{F^{\perp}}\tilde{dr}$, and $\tilde{F}\tilde{dr} = \tilde{F^{\parallel}}\tilde{dr} + \tilde{F^{\perp}}\tilde{dr}$ which imply the lemma. We record that $\tilde{F} = \sum_{i<j} F_{ij}c_i c_j$, and $\tilde{F^{\parallel}}\tilde{dr} = c_i F_{ri}$, where c_i or e_i comes from some orthonormal basis. □

Two other operators are of interest: let ∇_r denote the radial derivative of the spinor field. In the flat Euclidean case $\nabla_r = \frac{x_i}{|x|}\nabla_i$. Lastly, we consider the Dirac operator $D = c_i\nabla_i$.

Lemma 2.3.2. *(Commutation rules)*

$$\begin{cases} \nabla_r \tilde{dr} = \tilde{dr}\nabla_r \\ D\tilde{dr} + \tilde{dr}D = \frac{1-n}{|x|} - 2\nabla_r + O(r) \\ \langle Ds, t\rangle - \langle s, Dt\rangle = div(\langle c_i s, t\rangle e_i) \\ D\nabla_r - \nabla_r D = c_i F_{ir} + \frac{D}{|x|} - \frac{\tilde{dr}}{|x|}\nabla_r + O(1) \end{cases}$$

where the $O(r)$ and the $O(1)$ term is the non-Euclidean effect which vanishes in the Euclidean case, and in general does not depend on F.

Proof. The proof goes by using the geodesic coordinate expressions and applying Leibniz rules. The appearance of the $O(r)$ term comes from the Riemannian curvature effect, and the deviation of the Hessian of r from its Euclidean value. □

Corollary 2.3.3. (Some differential identity) Assume $Ds = 0$. Then

$$\frac{2-n}{|x|}\langle s, \nabla_r s\rangle - div(\langle c_k\tilde{dr}s, \nabla_r s\rangle e_k) - 2|\nabla_r s|^2 = \langle s, \tilde{F^{\parallel}}s\rangle + O(|s|^2 + r|s||\nabla_r s|) \quad (2.13)$$

We also record that $\langle c_k\tilde{dr}s, \nabla_r s\rangle e_k \cdot \nabla r = -\langle s, \nabla_r s\rangle$. We remark that the correction $O(|s|^2 + r|s||\nabla_r s|)$ is a non-Euclidean term whose strength is determined by the Riemannian curvature.

Proof. For cleaness let's present the Euclidean argument. Using the lemmas, $D\nabla_r s = \tilde{dr}\tilde{F^{\parallel}}s - \tilde{dr}\frac{\nabla_r s}{|x|}$, so $\langle\tilde{dr}s, D\nabla_r s\rangle = \langle s, \tilde{F^{\parallel}}s\rangle - \langle s, \frac{\nabla_r s}{|x|}\rangle$. But we also have

$$\begin{aligned} \langle\tilde{dr}s, D\nabla_r s\rangle &= \langle D\tilde{dr}s, \nabla_r s\rangle - div(\langle c_k\tilde{dr}s, \nabla_r s\rangle e_k) \\ &= \frac{1-n}{|x|}\langle s, \nabla_r s\rangle - div(\langle c_k\tilde{dr}s, \nabla_r s\rangle e_k) - 2|\nabla_r s|^2 \end{aligned}$$

Compare to get the desired identity. □

2.3.2 Integral identities

For the sake of clarity let's work in the Euclidean case. Notice 2.3.3 admits an integral version:

Lemma 2.3.4. *(Integral version)*

$$\int_{\partial B(r)} \frac{1}{2}\nabla_r |s|^2 = \int_{B(r)} \frac{n-2}{2|x|}\nabla_r |s|^2 + 2|\nabla_r s|^2 + \langle s, \tilde{F}^{\|} s\rangle \tag{2.14}$$

Proof. We compute

$$\begin{aligned}
\int_{\partial B(r)} \frac{1}{2}\nabla_r |s|^2 &= \int_{\partial B(r)} \langle s, \nabla_r s\rangle \\
&= \int_{\partial B(r)} -\langle c_k \tilde{dr} s, \nabla_r s\rangle e_k \cdot \nabla r \\
&= \int_{B(r)} div(-\langle c_k \tilde{dr} s, \nabla_r s\rangle e_k) \\
&= \int_{B(r)} \frac{n-2}{|x|}\langle s, \nabla_r s\rangle + 2|\nabla_r s|^2 + \langle s, \tilde{F}^{\|} s\rangle \\
&= \int_{B(r)} \frac{n-2}{2|x|}\nabla_r |s|^2 + 2|\nabla_r s|^2 + \langle s, \tilde{F}^{\|} s\rangle
\end{aligned}$$

□

This will be the basis of the subsequent integral identities. Let's first simplify some terms.

$$\begin{aligned}
&\int_{B(r)} \frac{n-2}{2|x|}\nabla_r |s|^2 dVol(x) \\
&= \frac{n-2}{2}\int_{\partial B(r)} \int_0^r \frac{d\xi}{\xi}\nabla_r |s|^2(\xi\theta) dA(r,\theta)(\frac{\xi}{r})^{n-1} \\
&= \frac{n-2}{2}\int_{\partial B(r)} r^{-1}|s|^2(r\theta) dA(r,\theta) - \frac{(n-2)^2}{2}\int_{B(r)} \frac{1}{\xi^2}|s|^2(\xi\theta) dVol
\end{aligned}$$

where the last step follows from integration by part along radial geodesics. With non-Euclidean effect included, the corrected RHS would be multiplied by $1 + O(r^2)$.

We then link this discussion to the derivative of the auxiliary function $g(t)$.

The RHS of 2.4 is:

$$
\begin{aligned}
\frac{dg}{dt} =& \frac{1}{t^{n-1}} \int_0^R rdr \int_{|x|=tr} \nabla_r |s|^2(x) dA(rt,\theta) \\
=& \frac{1}{t^{n-1}} \int_0^R 2rdr \int_{B(tr)} \frac{n-2}{2|x|} \nabla_r |s|^2 + 2|\nabla_r s|^2 + \langle s, \tilde{F^{\parallel}} s \rangle \\
=& \frac{1}{t^{n-1}} \int_0^R 2rdr \int_{B(tr)} 2|\nabla_r s|^2 + \langle s, \tilde{F^{\parallel}} s \rangle \\
&+ \frac{1}{t^{n-1}} [(n-2) \int_0^R rdr \int_{\partial B(tr)} (tr)^{-1} |s|^2(rt\theta) dA(tr,\theta) \\
&- (n-2)^2 \int_0^R rdr \int_{B(tr)} \frac{1}{\xi^2} |s|^2(\xi\theta) dVol] \\
=& \frac{1}{t^{n-1}} \int_0^R 2rdr \int_{B(tr)} 2|\nabla_r s|^2 + \langle s, \tilde{F^{\parallel}} s \rangle \\
&+ \frac{1}{t^{n+1}} [\frac{(n-2)n}{2} \int_{B(tR)} |s|^2 dVol - \frac{(n-2)^2}{2} \int_{B(tR)} \frac{(Rt)^2}{|x|^2} |s|^2(x) dVol]
\end{aligned}
\tag{2.15}
$$

The errors coming from non-Euclidean effects are of order $O((tR)^2|Riem|)$ compared to the main term.

We integrate in t from η to 1, and let η tend to 0. This leads to a cancellation effect:

$$
\begin{aligned}
&\int_0^1 dt \frac{1}{t^{n+1}} [\frac{(n-2)n}{2} \int_{B(tR)} |s|^2 dVol - \frac{(n-2)^2}{2} \int_{B(tR)} \frac{(Rt)^2}{|x|^2} |s|^2(x) dVol] \\
=& -\frac{(n-2)}{2} \int_{B(R)} |s|^2 dVol + \frac{(n-2)}{2} \int_{B(R)} \frac{(R)^2}{|x|^2} |s|^2(x) dVol \\
&+ \lim_{\eta\to 0} \frac{(n-2)}{2} \frac{1}{\eta^n} \int_{B(\eta R)} |s|^2 dVol - \frac{(n-2)}{2} \frac{1}{\eta^{n-2}} \int_{B(\eta R)} |s|^2 \frac{R^2}{|x|^2} dVol \\
=& -\frac{(n-2)}{2} \int_{B(R)} |s|^2 dVol + \frac{(n-2)}{2} \int_{B(R)} \frac{R^2}{|x|^2} |s|^2(x) dVol - |s(0)|^2 R^n \frac{\omega_{n-1}}{n} \\
=& -\frac{(n-2)}{2} \int_{B(R)} |s|^2 dVol + \frac{(n-2)}{2} \int_{B(R)} \frac{R^2}{|x|^2} |s|^2(x) dVol - g(0)
\end{aligned}
\tag{2.16}
$$

and the other terms are:

$$
\int_0^1 dt \frac{dg}{dt} = g(1) - g(0)
$$

and as before,

$$
\begin{aligned}
&\int_0^1 dt \frac{1}{t^{n-1}} \int_0^R 2rdr \int_{B(tr)} 2|\nabla_r s|^2 + \langle s, \tilde{F^{\parallel}} s \rangle \\
=& \int_{B(R)} [\frac{2R^2}{n(n-2)} (\frac{R}{|x|})^{n-2} - \frac{R^2}{n-2} + \frac{|x|^2}{n}][2|\nabla_r s|^2 + \langle s, \tilde{F^{\parallel}} s \rangle].
\end{aligned}
$$

This leads to the following integral identity in the Euclidean case:

Proposition 2.3.5. In the Euclidean case,

$$
\begin{aligned}
&-\frac{n}{2}\int_{B(R)}|s|^2 dVol + \frac{(n-2)}{2}\int_{B(R)}\frac{R^2}{|x|^2}|s|^2(x)dVol+ \\
&\int_{B(R)}[\frac{2R^2}{n(n-2)}(\frac{R}{|x|})^{n-2} - \frac{R^2}{n-2} + \frac{|x|^2}{n}][2|\nabla_r s|^2 + \langle s, \tilde{F^{\parallel}} s\rangle = 0.
\end{aligned}
\tag{2.17}
$$

We remark that the non-Euclidean effects are generally weaker than the main terms by order 2, so in particular can be controlled by the term

$$
C|Riem|(R^2\int_{B(R/2)}(\frac{R}{|x|})^{n-4}|\nabla_r s|^2 dVol + \int_{B(R)}|s|^2 dVol)
$$

using Hardy's inequalities 2.5.2.

We also record an inequality which follows immediately from the above proposition and Hardy's inequality. Here we need to notice that the coefficient in front of the gradient term is positive.

Corollary 2.3.6. (Radial gradient estimate from cancellation effects)

$$
\begin{aligned}
&\int_{B(R/2)} R^2(\frac{R}{|x|})^{n-2} 2|\nabla_r s|^2 dVol \\
&\leq C[\int_{B(R)} R^2(\frac{R}{|x|})^{n-2}|\langle s, \tilde{F^{\parallel}} s\rangle| dVol + \|s\|^2_{L^2(B(R))}]
\end{aligned}
\tag{2.18}
$$

The meaning of this inequality is that the radial gradient is well estimated if the most singular term in the radial curvature integral is controlled.

2.3.3 Consequences of Bianchi identity

There is a subtle differential identity satisfied by the curvature term which comes from the Bianchi identity. For the sake of clarity, let us do the computation in the Euclidean case. We let $X = F_{rj}e_j$, and let the tilde denote the Clifford multiplication operators associated to the tangent vectors, as before. We record that $\tilde{X}$ anticommutes with $\tilde{dr}$, and $\tilde{X} = \tilde{F^{\parallel}}\tilde{dr}$. We work with an orthonomal frame such that $\nabla e_i = 0$ at the point of interest.

Lemma 2.3.7. *(More commutation rules)*

$$
\begin{cases}
\nabla_k \tilde{F} - \tilde{F}\nabla_k = \tilde{\nabla_k F} = -\sum_{i\neq j}(\nabla_i F_{jk})c_i c_j \\
\tilde{\nabla_r F} = D\tilde{X} + \tilde{X}D + 2X_i\nabla_i + \sum_i \nabla_i F_{ri} \\
\nabla_r\langle s, \tilde{F}s\rangle + div(\langle c_i s, \tilde{X}s\rangle e_i) - div(\langle s, F_{ri}s\rangle e_i) = 2Re\langle\nabla_r s, \tilde{F}s\rangle + 2Re\langle s, X_i\nabla_i s\rangle
\end{cases}
$$

Proof. In the first identity, the first equality follows from the Leibniz rule, and the second from the Bianchi identity, which reads $\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0$.

For the second identity, we apply the Leibniz rule

$$\begin{aligned}
\nabla_r \tilde{F} &= \sum_{i \neq j} c_i \nabla_i (F_{rj}) c_j \\
&= \sum_{i,j} c_i \nabla_i (F_{rj}) c_j + \sum_i \tilde{\nabla}_i F_{ri} \\
&= \sum_i c_i \nabla_i \tilde{X} + \sum_i \tilde{\nabla}_i F_{ri} \\
&= D\tilde{X} - \sum_i c_i \tilde{X} \nabla_i + \sum_i \tilde{\nabla}_i F_{ri} \\
&= D\tilde{X} + \tilde{X} D + 2X_i \nabla_i + \sum_i \nabla_i \tilde{F}_{ri}.
\end{aligned}$$

We proceed to compute

$$\nabla_r \langle s, \tilde{F}s \rangle = 2Re\langle \nabla_r s, \tilde{F}s \rangle + \langle s, \nabla_r (\tilde{F}) s \rangle,$$

$$\langle s, \sum_i \nabla_i F_{ri} s \rangle = div(\langle s, F_{ri} s \rangle e_i) - 2\sqrt{-1} Im \langle s, X_i \nabla_i s \rangle,$$

$$\langle s, D\tilde{X}s \rangle = \langle Ds, \tilde{X}s \rangle - div(\langle c_i s, \tilde{X}s \rangle e_i).$$

Since $Ds = 0$, we have the third identity.

□

Proposition 2.3.8. (Integral form)

$$\begin{aligned}
\int_{\partial B(r)} \langle s, \tilde{F}s \rangle dA(r,\theta) - (n-1) \int_{B(r)} \frac{\langle s, \tilde{F}s \rangle}{|x|} dVol - \int_{\partial B(r)} \langle s, \tilde{F^{\|}} s \rangle \\
= \int_{B(r)} 2Re\langle \nabla_r s, \tilde{F}s \rangle + 2Re\langle s, X_i \nabla_i s \rangle dVol
\end{aligned} \tag{2.19}$$

Proof. We derive an integral version of 2.3.7. Using radial integration by part,

$$\int_{B(r)} \nabla_r \langle s, \tilde{F}s \rangle = \int_{\partial B(r)} \langle s, \tilde{F}s \rangle dA(r,\theta) - (n-1) \int_{B(r)} \frac{\langle s, \tilde{F}s \rangle}{|x|} dVol$$

We also compute

$$\int_{B(r)} div(\langle c_i s, \tilde{X}s \rangle e_i) = \int_{\partial B(r)} \langle c_i s, \tilde{X}s \rangle e_i \cdot \nabla r = \int_{\partial B(r)} \langle \tilde{d}rs, \tilde{X}s \rangle = -\int_{\partial B(r)} \langle s, \tilde{F^{\|}} s \rangle$$

$$\int_{B(r)} div(\langle s, F_{ri} s \rangle e_i) = \int_{\partial B(r)} \langle s, F_{rr} s \rangle = 0$$

□

Proposition 2.3.9. (Alternative integral version)

$$\begin{aligned}
\frac{-2}{n-3} \int_{B(R)} \frac{\langle s, \tilde{F}s \rangle}{|x|^{n-2}} dVol + \frac{n-1}{n-3} \int_{B(R)} \frac{\langle s, \tilde{F}s \rangle}{|x| R^{n-3}} dVol - \int_{B(R)} \frac{\langle s, \tilde{F^{\|}} s \rangle}{|x|^{n-2}} dVol \\
= \int_{B(R)} (2Re\langle \nabla_r s, \tilde{F}s \rangle + 2Re\langle s, X_i \nabla_i s \rangle) \frac{1}{n-3} \left(\frac{1}{|x|^{n-3}} - \frac{1}{R^{n-3}} \right) dVol
\end{aligned} \tag{2.20}$$

Proof. Integrate every term of the previous proposition by $\int_0^R \frac{dr}{r^{n-2}}$. We have in particular

$$\int_0^R \frac{dr}{r^{n-2}} \int_{\partial B(r)} \langle s, \tilde{F}s\rangle dA(r,\theta) = \int_{B(R)} \frac{\langle s, \tilde{F}s\rangle}{|x|^{n-2}} dVol$$

$$\int_0^R \frac{dr}{r^{n-2}} \int_{B(r)} \frac{\langle s, \tilde{F}s\rangle}{|x|} dVol = \frac{1}{n-3}\left[\int_{B(R)} \frac{\langle s, \tilde{F}s\rangle}{|x|^{n-2}} dVol - \int_{B(R)} \frac{\langle s, \tilde{F}s\rangle}{|x|R^{n-3}} dVol\right]$$

□

We can also immediately write down an estimate coming from the above proposition. We introduce a few maximal functions on curvature:

$$\begin{aligned} \left\|F^{\parallel}\right\|_2 &= \sup_{B(y,r)\subset B(2R)} \left(\int_{B(y,r)} \frac{|F^{\parallel}|^2(x)}{|x-y|^{n-4}} dx\right)^{1/2}, \\ \|F\|_2 &= \sup_{B(y,r)\subset B(2R)} \left(\int_{B(y,r)} \frac{|F|^2(x)}{|x-y|^{n-4}} dx\right)^{1/2}, \\ \left\|F^{\parallel}\right\|_1 &= \sup_{B(y,r)\subset B(2R)} \left(\int_{B(y,r)} \frac{|F^{\parallel}|(x)}{|x-y|^{n-2}} dx\right). \end{aligned} \tag{2.21}$$

The next estimate in fact takes into account Riemannian effects.

Corollary 2.3.10. (Inequality from Bianchi identity)

$$\begin{aligned} &\left|\frac{2}{n-3}\int_{B(R)} \frac{\langle s, \tilde{F}s\rangle}{|x|^{n-2}} dVol + \int_{B(R)} \frac{\langle s, \tilde{F^{\parallel}}s\rangle}{|x|^{n-2}} dVol\right| \\ &\leq C\|s\|_{L^\infty(B(R))}\left[\left(\int_{B(R)} \frac{|\nabla_r s|^2}{|x|^{n-2}} dVol\right)^{1/2}\|F\|_2 + \right. \\ &\left.\left(\int_{B(R)} \frac{|\nabla s|^2}{|x|^{n-2}} dVol\right)^{1/2}\left\|F^{\parallel}\right\|_2 + \|F\|_2 \frac{\|s\|_{L^2(B(R))}}{R^{n/2}}\right] \end{aligned} \tag{2.22}$$

Proof. We observe

$$|2Re\langle\nabla_r s, \tilde{F}s\rangle + 2Re\langle s, X_i\nabla_i s\rangle| \leq \|s\|_{L^\infty(B(R))}(2|\nabla_r s||F| + 2|\nabla s||F^{\parallel}|)$$

so the integral on the RHS of the previous proposition can be estimated by an application of Cauchy-Schwarz inequality.

We also notice all the Riemannian corrections to the LHS, which are at least of order 2 milder than the main terms, are estimated in terms of

$$|Riem|R^2 \int_{B(R)} \frac{|\langle s, \tilde{F}s\rangle|}{|x|^{n-4}} dVol$$

which look exactly similar to the lower order terms. Such terms can be estimated by:

$$\begin{aligned} &\left(\int_{B(R)} \frac{|\langle s, \tilde{F}s\rangle|}{|x|^{n-2-\alpha}} dVol\right) \leq \left(\int_{B(R)} \frac{|s|^2|F|^2}{|x|^{n-4}} dVol\right)^{1/2}\left(\int_{B(R)} \frac{|s|^2}{|x|^{n-2\alpha}} dVol\right)^{1/2} \\ &\leq \|F\|_2\|s\|_{L^\infty(B(R))}\left(\int_{B(R)} \frac{|s|^2}{|x|^{n-2\alpha}} dVol\right)^{1/2} \\ &\leq C\|F\|_2\|s\|_{L^\infty(B(R))}\left(\int_{B(R)} \frac{|\nabla_r s|^2}{|x|^{n-2-2\alpha}} dVol + \frac{1}{R^{n-2\alpha}}\|s\|^2_{L^2(B(R))}\right)^{1/2} \end{aligned} \tag{2.23}$$

where the last step uses Hardy's inequality. (cf. 2.5.2) □

2.4 Refined estimation of sup norm

We aim to bound $\|s\|_{L^\infty}$. The general strategy is the same as in 2.2. The idea is to assume a hypothetical bound (cf. 2.9), and then prove the constant to be independent of s under suitable smallness assumptions on the maximal functions of curvature, by improving this constant through a chain of estimates.

We consider the origin to save writing but similar estimates applies to any point in $B(2R)$.

The hypothetical bound reads: $|s|(y) \le \frac{M}{r^{n/2}} \|s\|_{L^2(B(y,r))}$ for $B(y,r) \subset B(2R)$. This means

$$\|s\|_{L^\infty(B(R))} \le \frac{M}{R^{n/2}} \|s\|_{L^2(B(2R))} \tag{2.24}$$

By the estimate on the curvature term 2.2.1, we have

$$R^2 \int_{B(R)} |\langle \tilde{F^{\parallel}} s, s\rangle| (\frac{R}{|x|})^{n-2} dVol \le M^2 \|s\|^2_{L^2(B(2R))} \|F^{\parallel}\|_1 \tag{2.25}$$

By the estimate on radial gradient 2.18, with 2.25, we get

$$\int_{B(R/2)} R^2 (\frac{R}{|x|})^{n-2} 2 |\nabla_r s|^2 dVol \le C \|s\|^2_{L^2(B(2R))} (1 + M^2 \|F^{\parallel}\|_1) \tag{2.26}$$

By the inequality from Bianchi identity 2.22, with the above estimates,

$$\begin{aligned} &|\frac{2}{n-3} \int_{B(R/2)} \frac{\langle s, \tilde{F} s\rangle}{|x|^{n-2}} dVol + \int_{B(R/2)} \frac{\langle s, \tilde{F^{\parallel}} s\rangle}{|x|^{n-2}} dVol| \\ &\le C \frac{M}{R^{n/2}} \|s\|_{L^2(B(2R))} [\frac{1}{R^{n/2}} \|s\|_{L^2(B(2R))} (1 + M \|F^{\parallel}\|_1^{1/2}) \|F\|_2 \\ &+ (\int_{B(R/2)} \frac{|\nabla s|^2}{|x|^{n-2}} dVol)^{1/2} \|F^{\parallel}\|_2 + \|F\|_2 \frac{\|s\|_{L^2(B(R/2))}}{R^{n/2}}] \end{aligned} \tag{2.27}$$

which, when combined with the estimate on the radial curvature term 2.25, gives the following estimate on the curvature forcing term

$$\begin{aligned} &| \int_{B(R/2)} \frac{\langle s, \tilde{F} s\rangle}{|x|^{n-2}} dVol| \\ &\le CM \|s\|_{L^2(B(2R))} \{ \frac{1}{R^n} \|s\|_{L^2(B(2R))} [(1 + M \|F^{\parallel}\|_1^{1/2}) \|F\|_2 + M \|F^{\parallel}\|_1] \\ &+ \frac{1}{R^{n/2}} (\int_{B(R/2)} \frac{|\nabla s|^2}{|x|^{n-2}} dVol)^{1/2} \|F^{\parallel}\|_2 \} \end{aligned} \tag{2.28}$$

This can be substituted into the subharmonic estimate 2.7, to achieve an estimate for $|s(0)|$ and the gradient integral term. Notice the lower order noises can be dealt with by 2.23.

$$
\begin{aligned}
&|s(0)|^2 Vol(B(R)) + \int_{B(R/2)} (\frac{R}{|x|})^{n-2} R^2 |\nabla s|^2(x) dVol \\
&\le C \|s\|^2_{L^2(B(R))} + CM\{\|s\|^2_{L^2(B(2R))} [(1 + M \|F^{\shortparallel}\|_1^{1/2}) \|F\|_2 + M \|F^{\shortparallel}\|_1] \\
&+ R^{n/2} (\int_{B(R/2)} \frac{|\nabla s|^2}{|x|^{n-2}} dVol)^{1/2} \|F^{\shortparallel}\|_2 \|s\|_{L^2(B(2R))}\}
\end{aligned} \tag{2.29}
$$

which by a simple application of Cauchy-Schwarz leads to

$$
\begin{aligned}
&|s(0)|^2 Vol(B(R)) \le C \|s\|^2_{L^2(B(2R))} \\
&\{1 + M[(1 + M \|F^{\shortparallel}\|_1^{1/2}) \|F\|_2 + M \|F^{\shortparallel}\|_1] + \|F^{\shortparallel}\|_2^2 M^2\}
\end{aligned} \tag{2.30}
$$

We emphasize again there is nothing special about the origin. This inequality allows us to improve the constant M; if M is already the optimal constant, then we must have

$$
M^2 \le C\{1 + M[(1 + M \|F^{\shortparallel}\|_1^{1/2}) \|F\|_2 + M \|F^{\shortparallel}\|_1] + \|F^{\shortparallel}\|_2^2 M^2\}
$$

We emphasize that C depends only on the Riemannian geometry, not on the connection. Now we can impose smallness assumptions on those curvature maximal functions. To make the above inequality interesting, there has to be some assumptions on the quadratic coefficients. We impose

$$
\|F^{\shortparallel}\|_2^2 < \frac{1}{4C}, \ \|F\|_2 \|F^{\shortparallel}\|_1^{1/2} < \frac{1}{4C}, \ \|F^{\shortparallel}\|_1 < \frac{1}{4C} \tag{2.31}
$$

The striking feature of this analysis is that $\|F\|_2$ itself does not have to be small. Then

$$
M \le C(1 + \|F\|_2) \tag{2.32}
$$

To summarize,

Theorem 2.4.1. *Under the bound on curvature 2.31, where the constant only depends on Riemannian geometry, we have the sup bound,*

$$
|s|(0) \le \frac{C(1 + \|F\|_2)}{R^{n/2}} \|s\|_{L^2(B(R))} \tag{2.33}
$$

and the gradient bound

$$
\int_{B(R/2)} (\frac{R}{|x|})^{n-2} R^2 |\nabla s|^2(x) dVol \le C(1 + \|F\|_2^2) \|s\|^2_{L^2(B(R))} \tag{2.34}
$$

Remark. This result is still a tiny bit weaker than the optimal expectation, namely to derive the same kind of bounds assuming only critical Morrey bound on F and critical integral bound on the $F^{\shortparallel}$.

Remark. We want to emphasize again the philosophy of this refined estimate: we can achieve a bound without assuming the curvature itself to be small. Only certain components of curvature are required to be small. This remark seems particular relevant for the study of typical gauge theories in which the Yang-Mills equation admits a first order reduction, so curvature is small in certain directions.

2.5 Appendix: Hardy's inequality

Lemma 2.5.1. *(Variant of one dimensional Hardy's inequality) For $\alpha > 0$, f a C^1 function on $[0, \infty)$ compactly supported on $[0, R]$,*

$$\int_0^\infty f'^2 r^{\alpha+1} \geq (\alpha/2)^2 \int_0^\infty f^2 r^{\alpha-1} \tag{2.35}$$

$$\int_0^\infty f'^2 r dr \geq 1/4 \int_0^\infty f^2 r^{-1} |log(|r|/2R)|^{-2} dr \tag{2.36}$$

$$\int_0^R f'^2 r^{\alpha+1} \geq (\alpha/2)^2 \int_0^R f^2 r^{\alpha-1} - \frac{\alpha}{2} f^2(R) R^\alpha \tag{2.37}$$

Proof. For the first part,

$$\begin{aligned}\int_0^\infty f'^2 r^{\alpha+1} \int_0^\infty f^2 r^{\alpha-1} &\geq (\int_0^\infty f' f r^\alpha)^2 \\ &= (\alpha/2 \int_0^\infty f^2 r^{\alpha-1})^2\end{aligned}$$

For the other parts, the proof is similar. □

Corollary 2.5.2. In the Euclidean case,

$$\int_{B(R)} \frac{f^2}{|x|^{-\alpha+n}} dVol \leq \frac{4}{\alpha^2} \int_{B(R)} \frac{(\nabla_r f)^2}{|x|^{-\alpha+n+2}} dVol + \frac{2R^{\alpha-n+1}}{\alpha} \int_{\partial B(R)} f^2 dA(R, \theta)$$

Proof. This follows by applying the Hardy inequality to the radial geodesics. It also clearly generalises to Riemannian case with some new coefficients. □

Remark. If $f = |s|$, the boundary integral is controlled in terms of the L^2 bound on $|s|$ and $|\nabla_r s|$, using the trace inequality in one dimension. The general way we use the Hardy inequality is to control lower order correction to integrals of $|s|$, in terms of gradient integrals.

Chapter 3

Remarks on convergence

Since our general goal as described in chapter 1 involves the singular connection A, let us remark on some implications of the a priori estimates on the convergence behaviour of Dirac fields $s_{(i)}$ on $B(2R)$, attached to a sequence of smooth connections $A_{(i)}$ converging to the singular one. This discussion has the nature of a compactness theory.

Assumptions: For definiteness, let us say $A_{(i)}$ tends to A in the L_1^2 sense. We impose some uniform bounds on the curvatures of the sequence of connections: we demand the small curvature bounds in the assumptions of the main estimates 2.2.2 or 2.4.1 to hold uniformly for $A_{(i)}$. Morever, to apply the dominated convergence theorem, we want the magnitudes of the curvature $|F_{(i)}|$ to be uniformly dominated by some scalar function f, such that $\|f\|_1 < \infty$. For the continuity issue discussion, let us assume also that as r tends to 0, there is the uniform convergence

$$\sup_{y \in B(2R)} \int_{B(r)} \frac{f(x)}{|x-y|^{n-2}} dVol(x) \to 0$$

3.1 Convergence issue

Because our estimates imply L_1^2 uniform control on $|s_{(i)}|$, by standard compactness result, we can assume by passing to subsquence $|s_{(i)}|$ converges strongly in $L^2(B(2R-\epsilon))$; this convergence does not use any convergence assumption on $A_{(i)}$. The general principle following this argument is: any zeroth order numerical gauge invariant quantity satisfies a **compactness result**.

On the contrary, to see that the sections $s_{(i)}$ subconverges to a section s in the L^2 sense, one needs the assumption that $A_{(i)}$ tends to A in the L^2 sense. The argument is similar, with the crucial difference that the gradient depends on the connection, and to run this compactness argument one needs to work with the fixed connection A. It is simple to see from the uniform estimates that the weak limit s in $L_1^2(B(2R-\epsilon))$ satisfies the **Dirac equation** for the limit connection, which makes sense because s is bounded and the connection matrix is L^1.

3.2 Non-collapsing of norms

Next we justify why the norms will not collapse. For this purpose, we need to recall that 2.7 is actually an equality, up to some Riemannian corrections one can explicitly write down. This is becasue the differential inequality we started with is actually an equality. In the Euclidean case, this reads:

$$\|s_{(i)}\|^2_{L^2(B(R))} = \int_{B(R)} [\frac{1}{n(n-2)}(\frac{R}{r})^{n-2}R^2 - \frac{1}{2(n-2)}R^2 + \frac{1}{2n}r^2]$$
$$[2|\nabla s_{(i)}|^2(r\theta) + 2\langle \tilde{F_{(i)}} s_{(i)}, s_{(i)}\rangle]dVol + |s_{(i)}(0)|^2 Vol(B(R)) \tag{3.1}$$

This is essentially the Green's formula for the solution of the Dirichlet problem of the Laplace equation. The same formula is also satisfied for the limit, since the Dirac equation holds for the limit. In the singular situation, the meaning of $|s|$ is given by

$$|s|^2(0) = \lim_{r\to 0} \frac{\int_{B(r)} |s|^2 dVol}{Vol(B(r))}$$

The Riemannian effects are less singular than the main terms, so we can safely ignore them.

Notice the integrals involving curvature will all converge by a simple application of the Lebesgue dominated convergence theorem, using the fact that L^2 convergence implies almost everywhere convergence.

Now the point is that, the gradient integral and the magnitude at the origin cannot increase in the limit, by standard measure theory. So the RHS of the Green's formula can only decrease in the limit; but the LHS converges to the limiting case, so the norm collapsing simply cannot happen in the interior, *i.e.* ,

$$|s_{(i)}(0)| \to |s|(0)$$
$$\int_{B(R/2)} (\frac{R}{r})^{n-2} |\nabla s_{(i)}| dVol \to \int_{B(R/2)} (\frac{R}{r})^{n-2} |\nabla s| dVol$$

This argument works for any interior balls; it essentially shows:

Theorem 3.2.1. *Under the standing assumptions, the magnitude of the Dirac fields converges pointwise, and the gradient integrals converge to the limiting case in any interior ball.*

This means the only possible mechanism of norm collapsing is for the L^2 mass of the Dirac fields to escape to the boundary sphere.

3.3 Continuity issue

Consider the Euclidean case. We address the continuity issue of $|s|$ using the Green's formula 3.1, applied to the limit s. This gives an integral representation of the central magnitude of s. Notice that the dominated convergence theorem easily implies the continuity of all but the most singular integrals. The slightly technical assumption ensures that the most singular integral involving the curvature also satisfies continuity. Therefore, the quantity

$$|s|^2(y)Vol(B(R)) + \int_{B(y,R)} \frac{2}{n(n-2)}(\frac{R}{|x-y|})^{n-2}R^2|\nabla s|^2(x)dVol(x)$$

is continuous in y. Now using Fatou's lemma, the gradient integral term is lower semicontinuous, so $|s|$ is upper semicontinuous.

The principle is general: once we have a Green's representation of a zeroth order numerical quantity, we can prove a version of continuity for that quantity. The relevant Green's representation can be obtained by a more refined application of the Weitzenbock formula, as follows. Work in the Euclidean case; let η be any parallel unit spinor field. Then by Weitzenbock formula,

$$-\Delta|\langle\eta, s\rangle|^2 = \nabla_i\nabla_i\langle s|\eta^\dagger\eta|s\rangle = 2|\langle\eta,\nabla s\rangle|^2 + 2Re\langle\tilde{F}s|\eta^\dagger\eta|s\rangle$$

So the analogue of 3.1 is:

$$\begin{aligned}\|\langle\eta, s\rangle\|^2_{L^2(B(R))} = \int_{B(R)}&[\frac{1}{n(n-2)}(\frac{R}{r})^{n-2}R^2 - \frac{1}{2(n-2)}R^2 + \frac{1}{2n}r^2]\\ &[2|\langle\eta,\nabla s\rangle|^2(r\theta) + 2\langle\tilde{F}s|\eta^\dagger\eta|s\rangle]dVol + |\langle\eta, s(0)\rangle|^2Vol(B(R)).\end{aligned} \tag{3.2}$$

Then the same argument as before says the quantity

$$|\langle\eta, s(y)\rangle|^2Vol(B(R)) + \int_{B(R)}[\frac{2}{n(n-2)}(\frac{R}{|x-y|})^{n-2}R^2|\langle\eta,\nabla s\rangle|^2(x)dVol$$

is continuous. Hence the same argument says

Proposition 3.3.1. The directional components of s, *i.e.* $\langle\eta, s(y)\rangle$, have upper semicontinuous modulus.

Morever, where $|s|$ is continuous, the modulus of the components cannot jump by semicontinuity, so they have to be continuous as well.

Here the intuition is: as far as the Weitzenbock formula is concerned, for weak bundle curvature, the different spin directions of the Dirac fields will decouple. This is the analogous statement for the Cauchy Riemann equation, where the real and imaginary parts of the function independently satisfy the Laplace equation.

Remark. Can we prove continuity for $|s|$?

Remark. The continuity of s is not a gauge invariant statement if we allow for highly singular gauge transforms. So we generally do not expect it to be true.

Chapter 4

Construction of Dirac fields

Now that we have the a priori estimates and a reasonable compactness theory, the main difficulty left to achieve the general goal as stated in chapter 1 is to find Dirac fields, defined on a ball of definite size, with the following conditions:

1. The Dirac field s is approximately a pure tensor field s' of unit norm, with the spinor S component being almost parallel, and the E component having almost constant norm.
2. The gradient of s is small with estimates.

We are allowed to work only with smooth connections satisfying the small curvature assumptions in the previous chapter. For clarity let us also work with the Euclidean metric. For the purpose of constructing such fields, we use a variational strategy. The technical aspect seems to work better with the $O(n)$ bundle case, where the Dirac field lives in $E \otimes_{\mathbb{R}} S$.

The main output of this chapter is a rigidity theorem, which asserts that under small $\|F\|_1$ assumption, if the maximum value for the variational problem is approximately optimal, then we have satisfactory estimates for all quantities of interest. We can relate the rigidity assumption to the spectrum of an operator. Under small curvature assumption and a spectral assumption, we give a construction for the approximately parallel and orthonormal basis required in Chapter 1.

4.1 Variational strategy

Before we start let us observe that the two requirements are intimately related. We consider the Euclidean metric, and we assume η is a parallel unit spinor field, such that $c_i\eta$ are orthonormal. This is where the $O(n)$ bundle case is more convenient than the Hermitian bundle case. More precisely, in the stantard construction of the spin representation S in terms of the exterior algebra $\Lambda^*\mathbb{C}^{n/2}$, then for $\eta = 1$, $c_i\eta$ are orthonormal with respect to the inner product of the underlying real vector space of S. The odd dimensional case is similar.

In the Riemannian situation it is also easy to arrange that to hold with good approximation. The Euclidean assumption is convenient but not consequential.

Proposition 4.1.1. A Dirac field s which is a pure tensor field with spinor component η has to be parallel.

Proof. Let $s = \xi \otimes \eta$, then $Ds = \sum \nabla_i \xi \otimes c_i \eta$, so $Ds = 0$ implies $\nabla s = 0$. □

This suggests that we look for Dirac fields whose S component is close to a parallel spinor field. But we know how to characterise such fields from the last section of the first chapter. So we consider the functional on the space of $L^2(B(2R))$ integrable Dirac fields, with unit L^2 norm.

$$H_f(s) = \int_{B(2R)} f|\langle \eta, s\rangle|^2 dVol$$

where f is a weight function, $0 \leq f \leq 1$. We will specialise in due course.

Remark. There are plenty of Dirac fields on $B(2R)$ (without any norm control). This is because one can solve the forced Dirac equation

$$Ds = \rho$$

where ρ is some section of $E \otimes S$ supported outside the ball $B(2R)$.

Remark. If we assume the L^2 integral of s to be 1, then $h_f = \sup H_f$ is obviously bounded by 1, so the maximisation problem is well defined.

Now the strategy is:

1. Find a maximising Dirac field for the functional by considering a maximising sequence.

2. Justify that the maximisation procedure does not collapse the norm.

3. Look for special features of the maximiser, to prove the required estimates at least in some definite interior ball.

4.2 The maximisation problem

We have a good compactness theory of convergence for Dirac field in the interior. Here we just need it in the case of a smooth fixed connection, so the convergence is smooth in the interior. The main problem is that the maximising sequence of Dirac fields may concentrate on the boundary sphere. For this, we require the weight function f to vanish near the boundary of $B(2R)$.

Let $h_f = \sup\{H_f : Ds = 0, \|s\|_{L^2} = 1\}$. Clearly $h_f > 0$. Now take a sequence of Dirac fields with L^2 integral 1, with $H_f \to h_f$. Then we can assume this sequence to converge in the interior smoothly to a Dirac field s. Thus clearly $H_f(s) = h_f$. But $\|s\|_{L^2} \leq 1$, so by the definition of h_f, the maximum has to be achieved and the global L^2 norm cannot collapse. This means the maximisation problem attached to H_f is easily solved.

The critical point equation for the H_f problem is:

$$\int_{B(2R)} f\langle t|\eta^{\dagger}\eta|s\rangle dVol = 0$$

for any Dirac field t in $L^2(B(2R))$, such that $\int_{B(2R)}\langle t, s\rangle dVol = 0$. This means

Proposition 4.2.1. The section $f\langle\eta, s\rangle \otimes \eta - h_f s$ is orthogonal in $L^2(B(2R))$ to any L^2 integrable Dirac field.

For most parts to follow, we will specialise to f being the characteristic function of $B(R)$. This vanishes in the annulus $B(2R) \setminus B(R)$ so the previous argument works. This means

$$H_f(s) = \int_{B(R)} |\langle\eta, s\rangle|^2 dVol$$

4.2.1 Interpretation 1: A decomposition of L^2

We give a different interpretation of 4.2.1. For this, let $W_0^{2,1}(B(2R))$ denote the completion of compactly supported sections in L_1^2. We have

$$D : W_0^{2,1}(B(2R)) \to L^2(B(2R))$$

We are working with a smooth connection, so the image of D is closed. We have

$$L^2 = ImD \oplus (ImD)^{\perp}$$

But being in the orthogonal complement of the image is precisely the weak formulation of the Dirac equation. This means $(ImD)^{\perp}$ is the space of L^2 integrable Dirac fields.

Thus 4.2.1 can be reformulated as

Proposition 4.2.2. The maximiser is characterised by the conditions: there exsits a $W_0^{2,1}(B(2R))$ section ζ, such that

$$\begin{cases} Ds = 0 \\ D\zeta = \rho = f\langle\eta, s\rangle \otimes \eta - h_f s \end{cases} \tag{4.1}$$

Notice by elliptic regularity, ζ is smooth in the interior, wherever f is smooth. Near the boundary, f vanishes, so $D\zeta = -h_f s$, hence $D^2\zeta = 0$ is satisfied, which is an elliptic equation with smooth coefficients; combined with the boundary condition $\zeta = 0$, we get the smoothness of ζ up to boundary, and so must s.

This set of conditions seems very restrictive, and the hope is that one can derive interior estimates depending only on the norm control on F, strong enough to imply our requirements.

4.2.2 Interpretation 2: spectral problem

Instead of a maximisation problem, one can also reinterprete the critical point condition as arising from a related spectral problem.

First, we define the Dirac analogue of Hardy space to be

$$H^2_D(B(r)) = \{s \in L^2(B(r)) : Ds = 0\}$$

Abstractly these are Hilbert spaces, with the natural L^2 inner product. We clearly have a restriction map

$$Res : H^2_D(B(2R)) \to H^2_D(B(R))$$

The Hermitian form $\langle t|\eta^\dagger \eta|Res(s)\rangle$ on $H^2_D(B(2R))$ can be thought of a self adjoint linear map on $H^2_D(B(2R))$, which is clearly bounded and semi-positive. Let's denote this map by L. Then L is the composition of three maps:

$$L : H^2_D(B(2R)) \to H^2_D(B(R)) \to L^2(B(2R)) \to H^2_D(B(2R))$$

where the second map is the Clifford multiplication extended by 0 and the third map is the orthogonal projection.

We notice L is a compact operator, because the Restriction map is compact. This is just a special case of our interior compactness theory in the previous chapter, from the abstract viewpoint.

The variational problem is encoded in the spectrum of L; the eigenvalues take value in non-negative reals. In particular, the largest eigenvalue is the maximum value for our variational problem. Our critical point condition is a special case of the eigenvalue equation, in the case of the largest eigenvalue.

4.3 Upper bound on optimal constant

In the situation we have in mind, the connection is trivial, the Dirac field is a pure tensor field and its bundle and spinor part are both parallel tensor fields of constant length. In this situation, clearly

$$\|\langle \eta, s\rangle\|_{L^2(B(R))} = (\frac{1}{2})^{n/2} \|s\|_{L^2(B(2R))}$$

This means the expected optimal constant is $h_f = (\frac{1}{2})^n$. To bound the optimal constant from above requires a priori estimates; to put a lower bound requires essentially a constructive process, or alternatively some spectral information. In this section we aim to derive an upper bound. The main idea is a variant of our derivation of 2.2.2, namely that under weak curvature assumption $|s|^2$ is essentially subharmonic. We first state the result in the non-optimal form.

Proposition 4.3.1. (Upper bound on optimal constant) For every small $\epsilon > 0$, once the norm of the curvature $\|F\|$ satisfies a corresponding small bound, then $h_f < (\frac{1}{2})^{n/2} + \epsilon$.

To prove this, we start with the expression for the derivative of the average function g, given in 2.6. Instead of integrating from 0 to 1, we integrate from some number q to $2-\epsilon$, where ϵ is some small number. We may change it from line to line by multiplying some absolute constant. We assume the curvature bound $\|F\|_1 << \epsilon$.

The result of this integration is

$$g(2-\epsilon) - g(q) = \int_{B((2-\epsilon)R))} \lambda(|x|/R, q) R^2 [|\nabla s|^2 + \langle \tilde{F} s, s\rangle] dVol \tag{4.2}$$

where the function λ comes from some double integral calculation

$$\lambda(|x|/R, q) = \int_q^{2-\epsilon} \frac{dt}{t^{n+1}} \int_{\frac{|x|}{R}}^t 2\xi d\xi = \int_q^{2-\epsilon} \frac{dt}{t^{n+1}} (t^2 - (\frac{|x|}{R})^2) 1_{t > \frac{|x|}{R}}$$

In the special case of $q=0$, this is

$$\int_{\frac{|x|}{R}}^{2-\epsilon} [\frac{dt}{t^{n-1}} - (\frac{|x|}{R})^2 \frac{dt}{t^{n+1}}]$$

giving rise to the kind of familiar factors before the gradient term. Noice all the gradient coefficients are nonnegative. This means g is **monotone** up to some small errors, which we proceed to estimate.

Now we estimate the curvature term. Notice

$$\|s\|_{L^\infty(B((2-\epsilon)R))} \leq \frac{C}{R^{n/2}} \|s\|_{L^2(B(2R))}$$

where C depends on ϵ and may be very big; but the curvature integral is small because it receives a smallness factor $\|F\|_1$.

$$\int_{B((2-\epsilon)R))} \lambda(|x|/R, q) R^2 |\langle \tilde{F} s, s\rangle| dVol \leq C \|F\|_1 \|s\|^2_{L^2(B(2R))} \leq \epsilon \|s\|^2_{L^2(B(2R))}$$

We also have

$$g(2-\epsilon) \leq (\frac{2}{2-\epsilon})^n g(2)$$

which we may rewrite as $g(2-\epsilon) \leq (1+\epsilon) g(2)$. Combining the above estimates with the integral identity we can achieve the almost monotonicity description mentioned above.

We are interested in two special consequences:

4.3.1 Case 1: $q = 0$

For $q=0$, we get $g(0) \leq (1+\epsilon) g(2)$, or in other words,

$$Vol(B(2R)) |s|^2(0) \leq (1+\epsilon) \|s\|^2_{L^2(B(2R))}$$

This estimate may be applied to balls with centre within ϵ distance to the origin. The result is

Proposition 4.3.2. (Sharp upper bound near the origin) Let ϵ be a small number and C is some small absolute constant. For $y \in B(C\epsilon)$, then as long as $\|F\|_1$ is sufficiently small depending on ϵ, we have the estimate

$$Vol(B(2R)) |s|^2(y) \leq (1+\epsilon) \|s\|^2_{L^2(B(2R))} \tag{4.3}$$

4.3.2 Case 2: $q = 1$

For $q = 1$, we notice that the function λ has a lower bound: for $|x|/R < 3/2$,

$$\lambda(|x|/R, q = 1) \geq C > 0$$

Proposition 4.3.3. (Upper bound with monotonicity term)

$$CR^2 \int_{B(3R/2)} |\nabla s|^2 dVol + \|s\|^2_{L^2(B(R))} \leq \frac{1}{2^n}(1+\epsilon) \|s\|^2_{L^2(B(2R))} \tag{4.4}$$

In particular the upper bound on h_f follows, as promised.

Notice the information being lost in the upper bound estimate of h_f can be recovered by a rigidity result.

Corollary 4.3.4. (Rigidity lemma) If the optimal bound is approximately attained, *i.e.* ,

$$\|\langle \eta, s\rangle\|^2_{L^2(B(R))} \geq ((\frac{1}{2})^n - \epsilon) \|s\|^2_{L^2(B(2R))} \tag{4.5}$$

then

$$\|s\|^2_{L^2(B(R))} \geq ((\frac{1}{2})^n - \epsilon) \|s\|^2_{L^2(B(2R))} \tag{4.6}$$

$$\|s\|^2_{L^2(B(R))} - \|\langle \eta, s\rangle\|^2_{L^2(B(R))} \leq C\epsilon \|s\|^2_{L^2(B(R))} \tag{4.7}$$

and there is a gradient bound

$$R^2 \int_{B(3R/2)} |\nabla s|^2 dVol \leq C\epsilon \|s\|^2_{L^2(B(R))} \tag{4.8}$$

4.4 Almost pure-tensorial Dirac fields

We want to explain in a more quantitative fashion why the requirement of almost pure-tensorial fields imply the other conditions. The basic idea is already contained in the very elementary statement 4.1.1. The aim of this section is to prove:

Theorem 4.4.1. *(Rigidity theorem) Assume a Dirac field s on a Euclidean ball $B(2R)$ satisfies the consequences of the above rigidity lemma. Suppose also the curvature norm $\|F\|_1 << \epsilon$. Then on the ball $y \in B(R/4)$, we have the following estimates:*

1. *Gradient integral estimate*

$$\int_{B(y,5R/4)} (\frac{R}{|x-y|})^{n-2} R^2 |\nabla s|^2(x) \leq C\epsilon \|s\|^2_{L^2(B(2R))} \tag{4.9}$$

2. *Pointwise estimate for 'orthogonal spin components'*

$$|\langle \eta', s(y)\rangle|^2 Vol(B(R)) \leq C\epsilon \|s\|^2_{L^2(B(2R))} \tag{4.10}$$

where η' is any parallel spinor field of unit length, which is orthogonal pointwise to η.

3. Pointwise bound on the modulus of $|s|$:

$$| \|s\|^2_{L^2(B(y,R))} - |s(y)|^2 Vol(B(R))| \le C\epsilon \|s\|^2_{L^2(B(2R))} \tag{4.11}$$

Morever, there is a small neighbourhood near the origin, of definite size of order ϵR, on which the pointwise modulus of $|s|^2$ is approximately the average value

$$| \|s\|^2_{L^2(B(R))} - |s(y)|^2 Vol(B(R))| \le C\epsilon \|s\|^2_{L^2(B(2R))} \tag{4.12}$$

In this section we will stick with these assumptions. It is clear that they are motivated by the optimal constant problem. One can think of the main statement as a rigidity theorem.

4.4.1 Weak curvature implies decoupling

The starting point of this discussion is 3.2, which as we recall says the following:

$$\begin{aligned}\|\langle\eta', s\rangle\|^2_{L^2(B(R))} = \int_{B(R)} [\frac{1}{n(n-2)}(\frac{R}{r})^{n-2}R^2 - \frac{1}{2(n-2)}R^2 + \frac{1}{2n}r^2] \\ [2|\langle\eta', \nabla s\rangle|^2(r\theta) + 2\langle\tilde{F}s|\eta'^{\dagger}\eta'|s\rangle]dVol + |\langle\eta', s(0)\rangle|^2 Vol(B(R))\end{aligned} \tag{4.13}$$

where η' is any parallel unit length spinor field. For us now $\eta' \perp \eta$. As we remarked earlier, this comes from the Weitzenbock formula, which essentially decompose into independent equations for individual spin components; the only way different spin components talk to each other is through the curvature term, which is insignificant under the small curvature assumption. We do the estimates more formally:

We notice as before that the coefficient in front of the gradient term is positively bounded below, away from the boundary. Now since by the main estimates 2.2.2, the sup norm of s is bounded by

$$\|s\|_{L^\infty(B(R))} \le \frac{C}{R^{n/2}} \|s\|_{L^2(B(2R))}$$

Hence the curvature integral is estimated by $\|F\|_1 \frac{C}{R^{n/2}} \|s\|_{L^2(B(2R))}$. Plug this into the integral representation, one sees immediately

$$\begin{aligned}\int_{B(R/2)} (\frac{R}{|x|})^{n-2} R^2 |\langle\eta', \nabla s\rangle|^2(x) dVol + |\langle\eta', s(0)\rangle|^2 Vol(B(R)) \\ \le C\epsilon \|s\|^2_{L^2(B(2R))} + C\|\langle\eta', s\rangle\|^2_{L^2(B(R))} \le C\epsilon \|s\|^2_{L^2(B(2R))}\end{aligned}$$

where the last step is because almost all L^2 norm is absorbed into the η spin direction, so the η' component is almost zero.

Notice the origin is not very special; the same argument would give for every $y \in B(R/4)$,

$$\begin{aligned}\int_{B(y,R/2)} (\frac{R}{|x-y|})^{n-2} R^2 |\langle\eta', \nabla s\rangle|^2(x) dVol(x) + |\langle\eta', s(y)\rangle|^2 Vol(B(R)) \\ \le C\epsilon \|s\|^2_{L^2(B(2R))}.\end{aligned} \tag{4.14}$$

In particular the orthogonal spin components of the Dirac field have pointwise small estimates in the ball, as promised.

4.4.2 Almost optimal constant implies small gradient

Notice a consequence of the rigidity lemma says that the L^2 gradient is small in the large interior ball $B(3R/2)$. But we have controlled the singularity of the gradient integral in the inequality above; hence for $y \in B(R/4)$,

Lemma 4.4.2. *(Gradient estimate for orthogonal spin components)*

$$\int_{B(y,5R/4)} (\frac{R}{|x-y|})^{n-2} R^2 |\langle \eta', \nabla s\rangle|^2(x) \le C\epsilon \|s\|^2_{L^2(B(2R))}$$

4.4.3 Dirac equation controls all gradient components

We are now in the situation where integrals of spin components of the gradient $\langle \eta', \nabla s\rangle$ are well controlled, for $\eta' \perp \eta$. This means Ds is close to $\langle \eta, \nabla_i s\rangle c_i \eta$ in the appropriate integral sense. Thus $|Ds|^2$ is close to $|\langle \eta, \nabla_i s\rangle c_i \eta|^2 = |\langle \eta, \nabla s\rangle|^2$. Hence we have the gradient bound

$$\int_{B(y,5R/4)} (\frac{R}{|x-y|})^{n-2} R^2 |\langle \eta, \nabla s\rangle|^2(x) \le C\epsilon \|s\|^2_{L^2(B(2R))}$$

Combining with previous results, we get the integral gradient estimate 4.9 as promised. This is of course the rigidity version of 4.1.1.

4.4.4 Green's representation controls the modulus

Notice that once again applying the Green representation formula which compares the ball average with the central value of $|s|^2$, with the knowledge that the gradient integral and the curvature integral are both small, we find for $y \in B(R/4)$,

$$| \|s\|^2_{L^2(B(y,r))} - |s(y)|^2 Vol(B(r))| \le C\epsilon \|s\|^2_{L^2(B(2R))}$$

where say $r \le 5R/4$. This implies the pointwise modulus estimate as promised.

Notice the interior estimate

$$|s|_{L^\infty(B(3R/2))} \le \frac{C}{R^{n/2}} \|s\|_{L^2(B(R))}$$

implies that for $y \in B(\epsilon R)$, we have

$$| \|s\|^2_{L^2(B(y,R))} - \|s\|^2_{L^2(B(0,R))} | \le \epsilon C \frac{\text{Area}(\partial B(R))}{R^{n-1}} \|s\|^2_{L^2(B(2R))} \le C\epsilon \|s\|^2_{L^2(B(2R))}$$

Hence as promised

$$| \|s\|^2_{L^2(B(R))} - |s(y)|^2 Vol(B(R))| \le C\epsilon \|s\|^2_{L^2(B(2R))}$$

This completes the proof of our rigidity theorem.

Remark. The reason the modulus at the origin is given by the average is that the subharmonicity forces a kind of monotonicity and convexity on the ball averages of $|s|^2$. The monotonicity says the $|s|^2(0)$ cannot be larger than the average. The rigidity assumption plus convexity puts a lower bound on $|s|^2(0)$, by forcing the ball averages at different scales to be essentially constant.

4.5 Consequences

The upshot of the rigidity theorem is that if the optimal upper bound is approximately achieved, then all quantitities of interests have saisfactory bounds. So far I am not able to give an interesting curvature condition to ensure this rigidity assumption, without using some gauge fixing results; so let us just put this as an independent assumption. With the general aim in our preliminary chapter in mind, we state this in a slightly stronger way:

Spectral assumption: for the spectral problem attached to the maximisation problem, *cf.* 4.2.2, let us assume the largest $m = rk(E)$ eigenvalues λ_j, $j = 1, \ldots, m$ are approximately as large as possible, *i.e.* , $\lambda_j \geq (\frac{1}{2})^n - \epsilon$.

Corollary 4.5.1. (Approximately orthonormal parallel basis) In the Euclidean case, under the small energy assumption on $\|F\|_1$, and the spectral assumption, we have all the estimates coming from the rigidity theorem. Morever, assuming the L^2 orthonormality of the eigenfields s_j just in case some eigenvalues coincide, then s_j are approximately orthogonal pointwise in the ball $B(\epsilon)$ with C^0 estimates: for $i \neq j$, $y \in B(\epsilon)$,

$$|\langle s_i, s_j \rangle| \leq C\epsilon \, \|s\|^2_{L^2(B(2R))} \tag{4.15}$$

Remark. If the eigenvalues do not coincide, which is the generic behaviour, elementary linear algebra implies that the L^2 orthogonality of the eigenfields are automatic.

Proof. Clearly the only statement which requires proof is the orthogonality estimate. We may normalise to $|s_j|(0) = 1$ for all j. The rigidity lemma with the rigidity theorem imply

$$| \, \|s\|^2_{L^2(B(2R))} - |s(y)|^2 Vol(B(2R))| \leq C\epsilon \, \|s\|^2_{L^2(B(2R))}$$

for $y \in B(\epsilon)$ and s in the span of $s_1, \ldots, s_m$. Apply this to $s = s_i$, $s = s_j$, and $s = s_i + s_j$, and use the polarisation identity to see the orthogonality claim. □

Remark. If one has a good curvature condition to imply the spectral assumption, then we would have essentially fulfilled all our promises required for the main picture in the preliminary chapter.

Remark. A possible idea to approach the spectral assumption is to encode the spectrum of the relevant pseudodifferential operator into some zeta function, and obtain an asymptotic formula, with sufficiently strong remainder estimate.

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